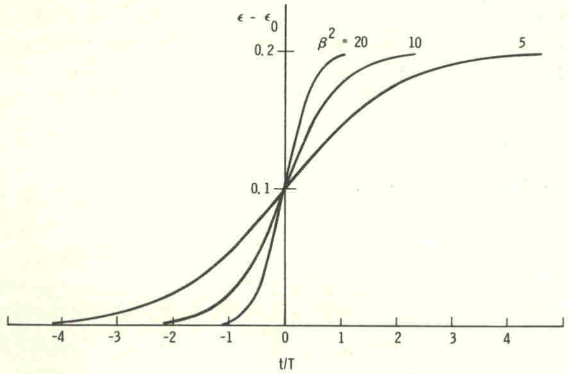

FIG. 4. Steady waveforms of amplitude $\epsilon_1 - \epsilon_0 = 0.2$ in a material collapsing according to Eq. (20). The effect of increasing curvature in the equilibrium stress-strain curve is illustrated.

are used merely to facilitate the parametric studies of Sec. IV.

B. Collapse Rules

In this section we address the question of how a porous solid collapses to an equilibrium state upon load application.

Let us consider a material point in a nonequilibrium state A and having the equilibrium stress-strain curve shown as the heavy solid line on Fig. 2. This figure has been drawn showing the state point A on the Rayleigh line, as it is in a steady wave, but we emphasize that the theory is intended to be applicable to any collapse process, and that A can refer to any state above the equilibrium curve. We will assume that the rate of collapse $\dot{\epsilon}$ of the material depends on its departure from equilibrium, and take as our measure of this departure the distance from A to the equilibrium state B at the same strain. With this assumption we have the collapse rule

$$\dot{\epsilon} = \phi_1 (\sigma - \sigma_E (\epsilon)) \quad (12)$$

One generalization of this collapse rule that seems appropriate is the introduction of strain-dependent weighting in the calculation of the collapse rate associated with a given departure from equilibrium. In this case it would take the form

$$\dot{\epsilon} = \phi (\epsilon, \sigma - \sigma_E (\epsilon)). \quad (13)$$

The necessity for this generalization is suggested by observations to be discussed subsequently.

The simplest special case of Eq. (12) that may be of interest is that in which the function ϕ_1 is linear and homogeneous:

$$\phi_1 = (1/\sigma^* T) [\sigma - \sigma_E (\epsilon)] \quad , \quad (14)$$

the case considered by Butcher. Clearly this function must involve a characteristic time and a characteristic stress in order that the dimensions of each member of Eq. (12) be the same. For the same reason, Eq. (13) must also involve one or more pairs of such constants.

The quasistatic collapse, under constant applied stress, of a material governed by Eq. (14) is readily found to be given implicitly by

$$t/T = \int_{\epsilon_A}^{\epsilon} \{\sigma^* [\sigma_A - \sigma_E (\lambda)]^{-1}\} \, d\lambda \quad , \quad (15)$$

and hence to be dependent on the form of the function $\sigma_E(\epsilon)$ as well as the constant $T\sigma^*$.

Before discussing wave-propagation problems involving Eq. (13), let us consider what conditions must be satisfied by the function ϕ in order that the material respond in a plausible fashion. It is clear that the collapse rate must vanish when the material is in equilibrium, so we must have $\phi(\epsilon, 0) \equiv 0$. Similarly, it seems plausible that the collapse rate should increase for increasing departure from equilibrium at any given strain, so we require that ϕ be a monotonic increasing function of its second argument. Finally, our intent in including the explicit dependence of $\dot{\epsilon}$ on ϵ was to accomodate the possibility that, for a given departure from equilibrium, the collapse rate would be greater at large strains than at small strains because the reduced void size in the first instance would be expected to lead to a smaller effective characteristic collapse time. To achieve this objective we require that ϕ also be a monotonic increasing function of its first argument when the second is held fixed. Surfaces ϕ meeting the above conditions slope upward as one proceeds in the direction of either increasing strain or overstress. In the special case of Eq. (12) the surface becomes a cylindrical sheet with generators parallel to the ϵ axis, and when the collapse rule is linear this cylinder becomes a plane. In these latter cases, of course, the collapse rule is completely represented by a single-valued curve in the $(\dot{\epsilon}, \sigma - \sigma_E)$ plane.

To see that Eq. (13) fits naturally into the usual theory[23] of rate-dependent constitutive equations as stated above, we note that it can be rewritten in the form $\sigma = \sigma_E(\epsilon) + \psi(\epsilon, \dot{\epsilon})$, where $\psi(\epsilon, 0) \equiv 0$, and $\psi(\epsilon, \dot{\epsilon})\, \dot{\epsilon} \geq 0$ by invoking the monotonicity conditions just discussed. In this form we see that it exhibits the usual decomposition of stress into equilibrium and nonequilibrium parts. The special case of Eq. (12) has the simple inverted form $\sigma = \sigma_E(\epsilon) + \phi_1^{-1}(\dot{\epsilon})$, which, in the linear case considered by Butcher, is further simplified to $\sigma = \sigma_E(\epsilon) + \sigma^* T \dot{\epsilon}$.

IV. STEADY-WAVE PROFILES

In this section we consider steady-wave propagation in materials governed by the collapse rule of Eq. (13) and any equilibrium curve $\sigma = \sigma_E(\epsilon)$ that is concave to the stress axis in the region of interest. The solution for general forms of the collapse rule and equilibrium curve is reduced to quadrature, and explicit closed-form results are given for special cases.

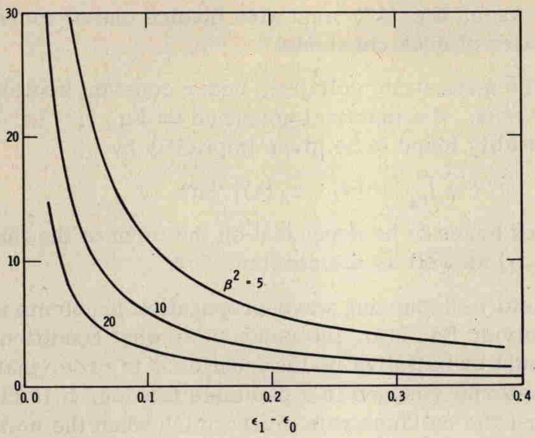

FIG. 5. Dependence of steady-wave rise time on amplitude according to Eq. (21).

Since the state far behind a propagating disturbance is one of equilibrium, $\sigma_1 = \sigma_E(\epsilon_1)$, we see from Eq. (10) that the speed of propagation of a steady wave is determined by the equilibrium stress-strain curve alone and is quite independent of the collapse rule. In general we have $V = \{(1/\rho_0)[\sigma_E(\epsilon_1) - \sigma_0](\epsilon_1 - \epsilon_0)^{-1}\}^{1/2}$. To solve for a steady waveform we simply substitute the first of Eqs. (6) and (7) into the collapse rule and integrate. In the general case we find that

$$\xi = -V \int_{(\epsilon_1+\epsilon_0)/2}^{\epsilon} \frac{d\lambda}{\phi(\lambda, \sigma_0 + \rho_0 V^2(\lambda - \epsilon_0) - \sigma_E(\lambda))}, \tag{16}$$

where the constant of integration has been chosen so that $\xi = 0$ at the half-amplitude point. Since $\phi(\epsilon, 0) = 0$ we see that the integrand is singular at both ϵ_0 and ϵ_1 so that the waveforms considered extend over the whole range $-\infty < \xi < \infty$. As a practical matter, however, we will see that the bulk of the variation is confined to a rather narrow region of space, or short interval of time. Since $\xi = X - Vt$, we can easily obtain the strain history at a fixed particle (in our examples we take $X = 0$; the waveform is independent of the choice). Stress and particle-velocity histories are obtained from the strain history by means of Eqs. (7) and (8) given in Sec. II.

In the special case where the stress-strain curve of Eq. (11) is used, Eq. (16) takes the simplified form

$$\xi = -V \int_{(\epsilon_1+\epsilon_0)/2}^{\epsilon} \frac{d\lambda}{\phi(\lambda, \rho_0 c_0^2 \beta^2(\epsilon_1 - \lambda)(\lambda - \epsilon_0))}, \tag{17}$$

where we now have

$$V = c_0 [1 - \beta^2(\epsilon_1 - \epsilon_0)]^{1/2} . \tag{18}$$

A. Linear Collapse Rule

As a specific example, let us determine the steady-wave profile implied by Eq. (17) when the linear collapse function (14) is used. Evaluation of the integral is routine and yields the result

$$\xi = -\frac{VT}{\beta^2(\epsilon_1 - \epsilon_0)} \log_e \left(\frac{\epsilon - \epsilon_0}{\epsilon_1 - \epsilon}\right), \tag{19}$$

where V is given by Eq. (18) and where we have taken $\sigma^* = \rho_0 c_0^2$. The strain history obtained from Eq. (19) is

$$\epsilon(t) = \epsilon_0 + \frac{(\epsilon_1 - \epsilon_0)\exp[\beta^2(\epsilon_1 - \epsilon_0)t/T]}{1 + \exp[\beta^2(\epsilon_1 - \epsilon_0)t/T]} . \tag{20}$$

Graphs of these waveforms as functions of amplitude are shown on Fig. 3 for the case $\beta^2 = 10$ and on Fig. 4 for the fixed amplitude $\epsilon_1 - \epsilon_0 = 0.2$ and varying values of β^2. The stress and particle-velocity histories can be obtained from Eq. (20) through the simple algebraic relations (7) and (8). Examination of Eq. (20) (and the figures) shows that the upper and lower halves of the waveforms are symmetrical. This symmetry is a property of waveforms governed by any collapse rule of the form of Eq. (12) if we use the quadratic stress-strain curve, but is not generally true otherwise, as is especially evident in the example of the locking solid to be discussed subsequently.

Since steady compressive disturbances propagate as shocks in the absence of dispersive tendencies, a simple measure of the influence of this latter effect is the steady-wave rise time. Various definitions of rise time are possible, but for simplicity, ease of experimental determination, and uniform applicability to various waveforms, we define the rise time $\mathscr{T}$ as that time interval required for the strain at a fixed particle to increase from $\epsilon_0 + 0.05(\epsilon_1 - \epsilon_0)$ to $\epsilon_0 + 0.95(\epsilon_1 - \epsilon_0)$. This is the same as the corresponding value for stress or particle velocity and is given by

$$\mathscr{T} = 5.889\, T[\beta^2(\epsilon_1 - \epsilon_0)]^{-1} \tag{21}$$

for the example at hand. We see that the rise time is proportional to the ratio of the characteristic time of the material to the nonlinear correction to the wave speed, and is thus determined by the relative importance of the tendencies toward dispersion and

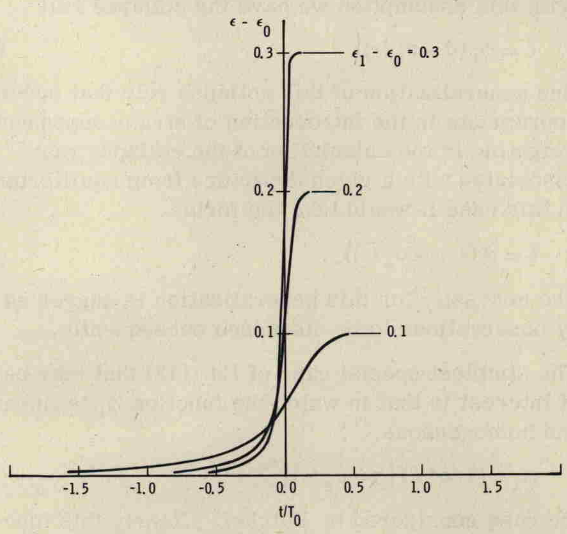

FIG. 6. Steady waveforms of various amplitudes in a material collapsing according to Eq. (24). We have taken $\beta^2 = 10$ and $\alpha^2 = 100$.